Week 7

Q1 Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the plane $W = \{(x,y,z) : x+2y-z=0\}$. Find the matrix of T with respect to the standard basis $\alpha = \{e_1, e_2, e_3\}$.

Solution:

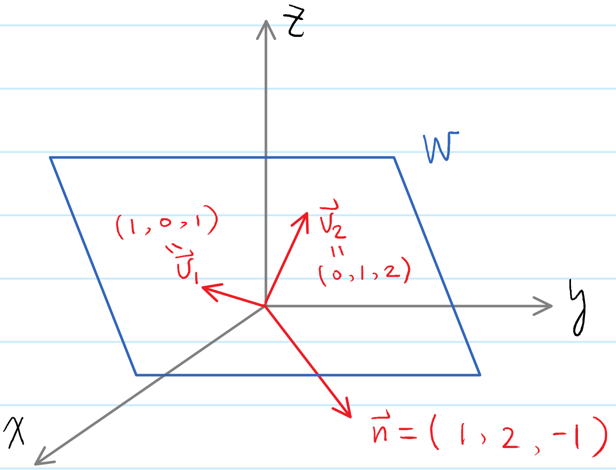
Note that the subspace W has a basis $\{\vec{v}_1, \vec{v}_2\}$

$\vec{v}_1 = (1,0,1)$, $\vec{v}_2 = (0,1,2)$

And the vector $\vec{n} = (1,2,-1)$ is orthogonal to the plane W.

So we have the following:

$$\begin{cases} T\vec{v}_1 = \vec{v}_1, \quad T\vec{v}_2 = \vec{v}_2 \\ T\vec{n} = 0 \end{cases}$$



So we have that

$$M(T,\beta,\beta) = \begin{bmatrix} M(T(\vec{v}_1),\beta) & M(T(\vec{v}_2),\beta) & M(T(\vec{n}),\beta) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

while $M(I,\beta,\alpha) = \begin{bmatrix} M(\vec{v}_1,\alpha) & M(\vec{v}_2,\alpha) & M(\vec{n},\alpha) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$

$$\Rightarrow M(T,\alpha,\alpha) = M(I,\beta,\alpha)\, M(T,\beta,\beta)\, M(I,\alpha,\beta)$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}^{-1}}_{= -\frac{1}{6}\begin{bmatrix} -5 & 2 & -1 \\ 2 & -2 & -2 \\ -1 & -2 & 1 \end{bmatrix}}$$

(by cofactors)

$$= \frac{1}{6}\begin{bmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

#

9 Give an example of a function $\varphi : \mathbf{C} \to \mathbf{C}$ such that

$$\varphi(w+z) = \varphi(w) + \varphi(z)$$

for all $w, z \in \mathbf{C}$ but φ is not linear. (Here $\mathbf{C}$ is thought of as a complex vector space.)

Solution: As $\mathbb{C}$ is a complex vector space, we define $\varphi(w) = \overline{w}$: conjugation of w.

Then for any scalar $\lambda \in \mathbb{C}$, any vector $w, z \in \mathbb{C}$,

$$\varphi(z+w) = \overline{z+w} = \overline{z} + \overline{w} = \varphi(z) + \varphi(w)$$

but $\varphi(\lambda z) = \overline{\lambda z} = \overline{\lambda}\,\overline{z} \neq \lambda \overline{z} = \lambda \varphi(z)$

if $\mathrm{Im}(\lambda) \neq 0$ and $z \neq 0$.

\#

8 Suppose V and W are finite-dimensional with $\dim V \geq \dim W \geq 2$. Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not surjective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

proof: Denote $\dim V = n$, $\dim W = m$, then $2 \leq m \leq n$.

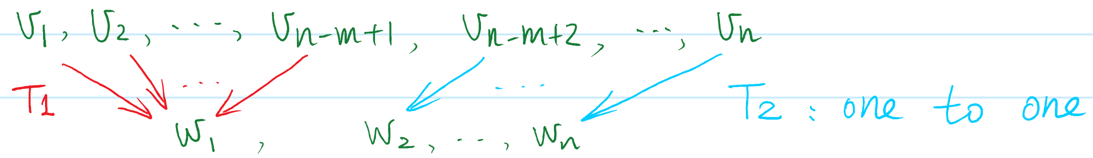
And denote $\mathbb{T} \triangleq \{T \in \mathcal{L}(V, W) : T \text{ is NOT surjective}\}$.

Assume V has a basis $\{v_1, \cdots, v_n\}$

W has a basis $\{w_1, \cdots, w_m\}$

Then we define two linear maps T_1 and T_2

satisfying $\begin{cases} \mathrm{Range}(T_1) = \mathrm{span}\{w_1\}, \quad \mathrm{Range}(T_2) = \mathrm{span}\{w_2, \cdots, w_m\}. \\ \mathrm{Range}(T_1 + T_2) = \mathrm{span}\{w_1, \cdots, w_n\} = W \end{cases}$



①. Define $\begin{cases} T_1(v_i) \triangleq w_1, & \text{for } 1 \leq i \leq n-m+1 \\ T_1(v_i) \triangleq 0, & \text{for } n-m+2 \leq i \leq n \\ T_1\left(\sum_{i=1}^{n} c_i v_i\right) \triangleq \sum_{i=1}^{n} c_i T v_i = \sum_{i=1}^{n-m+1} c_i w_1 \end{cases} \Rightarrow T_1 \in \mathbb{T}.$

②. Define $\begin{cases} T_2(v_i) \triangleq 0, & 1 \leq i \leq n-m+1 \\ T_2(v_i) \triangleq w_{i-n+m}, & n-m+2 \leq i \leq n \\ T_2\left(\sum_{i=1}^{n} c_i v_i\right) \triangleq \sum_{i=1}^{n} c_i T_2 v_i = \sum_{i=2}^{m} c_{i+n-m} w_i \end{cases} \Rightarrow T_2 \in \mathbb{T}$

Claim: $T_1 + T_2$ is surjective.

for $1 \leq i \leq n-m+1$, $(T_1 + T_2)(v_i) = T_1(v_i) + T_2(v_i) = w_1 + 0 = w_1$

for $n-m+2 \leq i \leq n$, $(T_1 + T_2)(v_i) = T_1(v_i) + T_2(v_i) = 0 + w_{i-n+m} = w_{i-n+m}$.

$\Rightarrow \mathrm{Range}(T_1 + T_2) = \mathrm{span}\{w_1, w_2, \cdots, w_m\} = W$.

\#

27 Suppose $p \in \mathcal{P}(\mathbf{R})$. Prove that there exists a polynomial $q \in \mathcal{P}(\mathbf{R})$ such that $5q'' + 3q' = p$.

[*This exercise can be done without linear algebra, but it's more fun to do it using linear algebra.*]

Proof: For any $p \in \mathcal{P}(\mathbb{R})$, denote $n = \deg(p(x)) \geq 0$. $p \in \mathcal{P}_n(\mathbb{R})$

Then we define $T: \mathcal{P}_{n+1}(\mathbb{R}) \to \mathcal{P}_n(\mathbb{R})$

via $Tq = 5q'' + 3q'$

Then T is linear.

$\Rightarrow \quad \dim(\text{Null } T) + \dim(\text{Range } T) = \dim(\mathcal{P}_{n+1}(\mathbb{R})) = n+2.$

It suffices to prove $\dim(\text{Range } T) = n+1$ (i.e. T is surjective)

which is equivalent to prove $\dim(\text{Null } T) = 1$.

In fact, if $q(x) = \sum_{i=0}^{n+1} a_i x^i \in \text{Null } T$

$\Rightarrow 5q'' + 3q' = 0$

while $q'(x) = \sum_{i=1}^{n+1} i \cdot a_i \cdot x^{i-1} = \sum_{k=0}^{n} (k+1) a_{k+1} x^k$

$\Rightarrow \quad q''(x) = \sum_{k=1}^{n} (k+1) k \, a_{k+1} x^{k-1} = \sum_{l=0}^{n-1} (l+2)(l+1) a_{l+2} x^l$

then $\sum_{l=0}^{n-1} 5(l+2)(l+1) a_{l+2} x^l + \sum_{k=0}^{n} 3(k+1) a_{k+1} x^k = 0$

$\Rightarrow \quad \sum_{l=0}^{n-1} \left[5(l+2)(l+1) a_{l+2} + 3(l+1) a_{l+1} \right] x^l + 3(n+1) a_{n+1} x^n = 0$

$$\Rightarrow \begin{cases} 3(n+1) \cdot a_{n+1} = 0 \\ 5(l+2)(l+1) a_{l+2} + 3(l+1) a_{l+1} = 0, \quad \forall \; 0 \leq l \leq n-1 \end{cases}$$

$\Rightarrow \quad a_{n+1} = a_n = \cdots = a_1 = 0.$

So $q(x) \equiv a_0 \quad \Rightarrow \quad \text{Null } T = \text{span}\{1\}$

$\Rightarrow \quad \dim(\text{Null } T) = 1.$

#

7 Suppose V and W are finite-dimensional. Let $v \in V$. Let

$$E = \{T \in \mathcal{L}(V, W) : Tv = 0\}.$$

(a) Show that E is a subspace of $\mathcal{L}(V, W)$.

(b) Suppose $v \neq 0$. What is $\dim E$?

Proof:

(a). Omitted.

(b). Assume V has a basis $\{v_1, \cdots, v_n\} \triangleq \alpha$

W has a basis $\{w_1, \cdots, w_m\} \triangleq \beta$

where without loss of generality, we choose $v_1 = v$. (See Note)

Note: $\exists$ subspace V_1 of V, s.t. $V = \mathrm{span}\{v\} \oplus V_1$ then choose a basis $\{v_2, \cdots, v_n\}$ of V_1, then $\{v, v_2, \cdots, v_n\}$ is a basis of V.

Thus for any $T \in E$,

$$M(T, \alpha, \beta) = \underbrace{\begin{bmatrix} M(T(v_1), \beta), & M(T(v_2), \beta), & \cdots, & M(T(v_n), \beta) \end{bmatrix}_{m \times n}}_{\text{columns}}$$

$$= \Big[0, \underbrace{M(T(v_2), \beta), \cdots, M(T(v_n), \beta)}_{m \times (n-1)\ \text{submatrix}}\Big] \text{, denoted by } S_M(T)$$

Then $S_M : E \to M_{m \times (n-1)}(\mathbb{R})$ is a linear map.

Claim: S_M is a bijection.

① Injection: If $S_M(T) = 0 \Rightarrow M(T, \alpha, \beta) = 0_{m \times n} \Rightarrow T = 0$

② Surjection: $\forall\ A = \begin{bmatrix} a_{11} & \cdots & a_{1,n-1} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{m,n-1} \end{bmatrix} \in M_{m \times (n-1)}(\mathbb{R})$.

then $\exists\ T \in \mathcal{L}(V, W)$, such that $M(T, \alpha, \beta) = [0, A]_{m \times n}$

Actually, define T on the basis α: see the constructions in Q2.

$$\begin{cases} T(v_1) = 0 \\ T(v_i) = \sum_{j=1}^{m} a_{j,i-1} w_j, \quad 2 \le i \le n. \end{cases}$$

and extend T linearly to the whole space V to make $T \in \mathcal{L}(V, W)$

$$\Updownarrow$$

$$M(T, \alpha, \beta) = [0 \ \ A]$$

that is $A = S_M(T)$

So $S_M : E \to M_{m \times (n-1)}(\mathbb{R})$ is an isomorphism

we have $\dim E = \dim(M_{m \times (n-1)}(\mathbb{R})) = m \cdot (n-1)$ #